Cayley's Theorem and Sofic Groups

Idea: We'd like to regard <u>all</u> finite groups as subgroups of "nicer" groups. For example, we could ask for all groups to be subgroups of $\mathbb{Z}_n$ for some n, but the fact that $\mathbb{Z}_n$ is abelian prohibits this.

Theorem: (Cayley) Let G be a **finite** group. Then $\exists$ $n \in \mathbb{N}$ such that G is isomorphic to a <u>subgroup</u> of S_n.

proof: Recall that S_n is merely all bijections on an n-element set. Set $\boxed{n = |G|}$.

We regard G as permutations on itself via left multiplication:

The injective homomorphism from G to $S_{|G|}$ will be

$$\boxed{\varphi(x) = L_x}$$

, where

$\forall\ y \in G$,

$$L_x(y) = x \cdot y.$$

Let's check that φ is an injective homomorphism!

φ injective: Suppose

$$\varphi(x) = \varphi(z)$$

for some $x, z \in G$.

Then choosing $y = e$,

$$x = x \cdot e = L_x(e) = \varphi(x)(e) = \varphi(y)(e)$$
$$= L_y(e)$$
$$= y \cdot e$$
$$= y$$

$\Rightarrow$ $x = y$, and φ is injective.

φ is a homomorphism

If $x, z \in G$. Then $\forall y \in G$,

$$\varphi(xz)(y) = L_{xz}\, y$$
$$= (xz)y$$
$$= x \cdot (zy)$$

$$= x \cdot L_z(y)$$

$$= L_x(L_z(y))$$

$$= (\ell(x) \circ \ell(z))(y)$$

The proof is complete by simply providing an identification of G with $\{1, 2, \ldots, |G|\}$.

For example, label the group elements $\{x_1, x_2, \ldots, x_{|G|}\} = G$ map $x_i \to i \quad \forall\ 1 \le i \le |G|$.

$\square$

Example 1: $(\mathbb{Z}_4)$ Set

$$[0] = 1$$

$$[1] = 2$$

$$[2] = 3$$

$$[3] = 4$$

in the identification.

Clearly $L_{[0]}$ is the identity permutation.

Since $\mathbb{Z}_4$ is cyclic, we only need to see what $L_{[1]}$ does.

$$L_{[1]}([0]) = [1] + [0] = [1]$$

$$L_{[1]}([1]) = [1] + [1] = [2]$$

$$L_{[1]}([2]) = [1] + [2] = [3]$$

$$L_{[1]}([3]) = [1] + [3] = [0]$$

So under the identification,

$L_{[1]}$ is the permutation

$$(1\,2\,3\,4).$$

$$L_{[2]} = (1234)(1234) = (13)(24)$$

$$L_{[3]} = (1\,4\,3\,2)$$

Note: $(4321) = (1432)$

So R_4 is isomorphic to

$$\langle (1\,2\,3\,4) \rangle$$

under this identification.

Everything is Matrices

To complete the picture, we can now regard every finite group as a subgroup of $GL_n(\mathbb{R})$ for some $n \in \mathbb{N}$ simply by identifying S_k as a subgroup $\forall$ $k \in \mathbb{N}$.

Write $\{e_1, e_2, \dots, e_n\}$ for the standard basis of $\mathbb{R}^n$.

If $\sigma \in S_n$, define

$$\varphi(\sigma) \in GL_n(\mathbb{R}),$$

$$\varphi(\sigma)\left(\sum_{i=1}^{n} a_i e_i\right)$$

$$= \sum_{i=1}^{n} a_i e_{\sigma(i)}$$

As a matrix, this is

$$\begin{bmatrix} e_{\sigma(1)} & e_{\sigma(2)} & \cdots & e_{\sigma(n)} \end{bmatrix}$$

Then φ will be an injective homomorphism from S_n into $GL_n(\mathbb{R})$!

Sofic Groups

(Gromov, 1999)

Length function on S_n $\quad \ell_{S_n}$ is defined by

$$\ell_{S_n}(\sigma) = \frac{\left|\{1 \le i \le n \mid \sigma(i) \ne i\}\right|}{n}$$

So $\ell_{S_n}(e) = 0$

$\ell_{S_n}(\sigma) = 1$ if σ is an n-cycle.

Definition: (sofic group) A countable discrete group G is said to be sofic if for every finite subset F of G and for all $\varepsilon>0$, there exists $n\in\mathbb{N}$ and $f_n: G\to S_n$ not necessarily a homomorphism such that

$$f_n(e_G)=e_{S_n} \quad \text{and}$$

1) $$l_{S_n}\left(f_n(xy)f_n(y)^{-1}f_n(x)^{-1}\right)<\varepsilon \quad \forall\, x,y\in F$$

(on F, f_n is "almost" a homomorphism)

2) $\left| \ell_{s_n}(\varphi(x)) - 1 \right| < \varepsilon$

$\forall \; x \in F \setminus \{e\}$

(f_n is not the trivial homomorphism)

Observations :

1) Finite groups are all sofic.

2) $\mathbb{Z}$ is sofic

3) The discrete Heisenberg group G,

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{Z} \right\}$$

is thought to not be sofic.

As of this writing, whether there is a non-sofic countable discrete group is an open problem!